Fuzzy Implication Methods in Fuzzy Logic

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Abstract

The Choice of fuzzy implication methods is a significant problem in the theoretical development of fuzzy set and approximate reasoning. Many operation methods on the fuzzy implication were presented. In this research, first we analyze the truth values of the traditional two valued logical implication using sets theory, propose a novel truth table of the two valued logical implication, then acquire the simplification formulas of the expanded two-valued logical implication using the Karnaugh-map tool; Second we explore the relation of the expanded two valued logical implication and existent fuzzy implication operation methods, subsequently we propose two novel operation methods on the fuzzy implication based on the relation. We validate these two methods with mathematical deduction as well as a concrete example. At last, we compare the superiority of various fuzzy implication methods.

1. Introduction

Since the concept of fuzzy sets was introduced by Zadeh[1], various methods of approximate reasoning have been presented and used as formal mathematical tools for reasoning under vagueness[2,3]. As is well known, approximate reasoning has become a theoretical basis and an important method for the design and analysis of fuzzy controller, and it has found a considerable number of successful industrial applications in some fields such as intelligent control [4]. Nevertheless, there are still some serious problems regarding the mathematical foundation of fuzzy logic to be solved and they deserve an intensive research [5], One of such problems is the choice of implication methods because the implication is one of the major connectives in any logical system, and it has very serious influence on the performance of the systems in which fuzzy logic technique is employed.

Zadeh did earlier researches on fuzzy implication [6, 7]. Mamdani and Larsen also contributed to this issue [8, 9]. Some recent progress on this problem could be referred in [10,11]. We explored their research results and found some significant rules, then utilize these rules to conduct our further investigation upon fuzzy implication.

The paper is organized as follows, the first section analyzes the implication of two valued logic, and we propose an expanded truth value table of two valued logical implication, which includes four cases instead of one case. In the second section, we utilize the tool of Karnaugh maps to simplify two valued logical implication formulas in the four cases. The third section creates the relation between expanded two value logical implication formulas and current existent fuzzy implication formulas. Along the relation, we propose two novel fuzzy implication methods, in the forth section, we introduce approximate reasoning and give an example to validate our methods, then we compared various implication methods. The fifth section is our conclusion.

2. Two Valued Logical Implication

We have ever learned the truth value table of implication in the following way. When the truth value of A is 0, regardless the truth value of B, the truth value of $A \rightarrow B$ is 1. In fact, we argue on this traditional rule. Now we analyze the implication from the angle of sets.

A, B are expressed in sets, we recognize the implication using Figure 1, X is a universal set here.

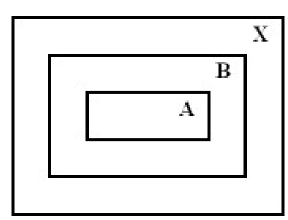

Figure 1. Sets of implication

This figure shows that if a point x locates in the set A, then x must locates in the set B, the value of $A \subset B$

must be the TRUE, that is 1. If the point x locates in the set A, and it does not locate in the set B, the value of $A\subset B$ must be the FALSE, that is 0, because this case could not exist. However, when the point x does not locate in the set A, and it locates in the set B, then the value of $A\subset B$ may be TURE or FALSE. We summarize these cases in the Table 1.

Table 1. Truth value table of $A\subset B$

A	B	$A\subset B$
Point x locates in set A (1)	Point x locates in set B (1)	TRUE（1）
Point x locates in set A (1)	Point x does not locate in set B (0)	FALSE（0）
Point x does not locate in set A (0)	Point x locates in set B (1)	MAYBE TRUE（1）/MAYBE FALSE（0）
Point x does not locate in set A (0)	Point x does not locate in set B (0)	MAYBE TRUE（1）/MAYBE FALSE（0）

Therefore, the truth values of $A\to B$ have four cases. See Table 2.

Table 2. Truth value table of $A\to B$

A	B	$A\to B$			
		①	②	③	④
1	1	1	1	1	1
1	0	0	0	0	0
0	1	1	1	0	0
0	0	1	0	1	0

3. Simplifying Implication Formula

The Karnaugh map, which is presented in 1950's by Karnaugh who worked in Bell Laboratory, is a graph which expresses logical function. The follow is the process of transforming a two value logical implication formula $A\to B$ to a two value Karnaugh map.

(1) First, the proposition A, B, $\overline{A}$ and $\overline{B}$ are separately written on the edge of the Karnaugh map.
(2) Second, an original code such as A indicates that the truth value of a proposition is 1; and a complement code such as $\overline{A}$ indicates that the truth value of a proposition is 0.
(3) Third, the truth value of the implication $A\to B$ is written in the corresponding grids.
(4) Forth, pick out items whose truth values are 1 in these grids, and then we get a disjunction of these items. The simplification of $A\to B$ is this disjunction.
(5) Fifth, pick out items whose truth values are 0 in these grids, and then we get a conjunction of these items. The simplification of A→B is the complement of this conjunction.

For example, the simplification of the first case with Karnaugh map is shown as follow.

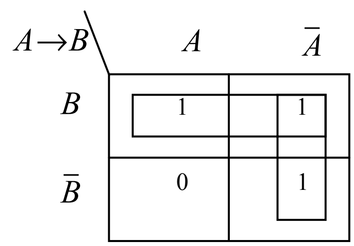

Figure 2. Simplifying the first case

If we do not take account of the aggregation of items, the disjunction is $A\to B=AB+\overline{A}B+\overline{A}\overline{B}$.

In this Karnaugh map, items that are neighboring and have the same value can be aggregated and form a maximum neighboring area, see Figure 2. The horizontal rectangle including AB and $\overline{A}B$ is B. The overlapped item such as '1' in the figure can be used repeatedly. Therefore, in this Karnaugh map, A→B= B+$\overline{A}$, AB+$\overline{A}$, $\overline{A}\overline{B}$+B, in addition, A→B= $\overline{A\overline{B}}$.

Similarly with the above case, we simplify the second, third and forth case in the expanded truth table using Karnaugh map.

For the second case, if we do not take account of aggregation then we quickly get $A\to B=AB+\overline{A}B$ or $\overline{A\overline{B}+\overline{A}\overline{B}}$.Else we get $A\to B=B$.

Similarly with above cases, in the third case, we get $A\to B=AB+\overline{A}\overline{B}$. And in the forth case, we get $A\to B=AB$ or $\overline{\overline{B}+\overline{A}}$.

We summarize the above results which were simplified by Karnaugh map in the Table 3.

Table 3. Simplification of $A \rightarrow B$

$A \rightarrow B$			
①	②	③	④
$B+\overline{A}$ $AB+\overline{A}$ $\overline{A}\overline{B}+B$ $\overline{A\overline{B}}$	$AB+\overline{A}B$ $\overline{A\overline{B}+\overline{A}\overline{B}}$ B	$AB+\overline{A}\overline{B}$ $\overline{A\overline{B}+\overline{A}B}$	AB $\overline{\overline{B}+\overline{A}}$

4. Fuzzy Implication

In this section, we begin our research from the simplifications above. We analyze the relation of existent fuzzy implication methods and these simplifications. Based on these relations, we present two different methods.

Mamdani presented a fuzzy implication method which we call minimum operation method[10, 12], that is

$$R_c = \underset{\sim}{A} \rightarrow \underset{\sim}{B} = \int_{X\times Y} (\mu_A(x) \wedge \mu_B(y))/(x,y)$$

.

He proved the first deduction of the forth case.

Zadeh presented a fuzzy implication method which we call maximum and minimum operation method[11, 13] and gave the prove, that is

$$R_m = \underset{\sim}{A} \rightarrow \underset{\sim}{B} = \int_{X\times Y} (\mu_A(x) \wedge \mu_B(y)) \vee (1-\mu_A(x))/(x,y).$$

He presented another fuzzy implication method which we call Boolean operation method, that is

$$R_b = \underset{\sim}{A} \rightarrow \underset{\sim}{B} = \int_{X\times Y} (1-\mu_A(x)) \vee \mu_B(y)/(x,y).$$

This method proves the fist deduction of the first case.

Larsen presented a fuzzy implication method which we call product operation method[13], that is

$$R_p = \underset{\sim}{A} \rightarrow \underset{\sim}{B} = \int_{X\times Y} \mu_A(x)\mu_B(y)/(x,y).$$

This method proved the second deduction of the forth case.

Based on the above deduction in the second and third case, we introduce two novel operation methods. One is

$$R = \underset{\sim}{A} \rightarrow \underset{\sim}{B} = \int_{X\times Y} (\mu_A(x) \wedge \mu_B(y)) \vee ((1-\mu_A(x)) \wedge \mu_B(y))/(x,y) = \int_{X\times Y} \mu_B(y)/(x,y)$$

The other is

$$R = \underset{\sim}{A} \rightarrow \underset{\sim}{B} = \int_{X\times Y} [\mu_A(x) \wedge \mu_B(y)] \vee [(1-\mu_A(x)) \wedge (1-\mu_B(y))]/(x,y)$$

.

We name these two implication methods R_B and R_X respectively.

We now prove the rationality of these two formulas. In the first formula, because $\mu_B(y) \in$ （0, 1）, it is reasonable. In the second formula, $1-A(x)$, $1-\mu_B(y) \in$ （0,1）,so $(1-A(x)) \wedge (1-\mu_B(y)) \in$ （0,1）, and $A(x) \wedge B(y) \in$ （0,1）, therefore $A(x) \wedge B(y)] \vee [(1-A(x)) \wedge (1-\mu_B(y)) \in$ （0, 1）, it is reasonable.

We utilize the tool of Karnaugh maps to simplify these expanded two valued logic implication, subsequently create the relation of the expanded two valued logical implication formulas and current existent fuzzy logical implication formulas. Along this idea, we present two novel fuzzy logical implication formulas. In next section, we validate the soundness of these two formulas through an example. The two valued logic is the foundation of researches on fuzzy logic. From the process of above deductions, we can conclude that our research approach is more intuitionistic than anterior research work.

5. Approximate Reasoning

The mathematical form of general approximate reasoning is denoted as follow, given a major premise which is "if x is A then y is B", that is $\underset{\sim}{A} \rightarrow \underset{\sim}{B}$,and a minor premise which is "x is A'". We deduce the conclusion of y is B' . The process is like $A' \rightarrow (\underset{\sim}{R} = \underset{\sim}{A} \rightarrow \underset{\sim}{B}) \rightarrow B'$. The y is $\underset{\sim}{B'} = \underset{\sim}{A'} \circ (A \rightarrow B) = \underset{\sim}{A'} \circ \underset{\sim}{R}$, here " $\circ$ " is a composition operator, "$\underset{\sim}{R}$" is an implication relation. We give an example as follow.

Adjusting the temperature of a furnace has following experience: If the temperature is low, then inflict high voltage. Now if the temperature is a little low, how does the voltage should be inflicted?

To solve this question, we first create a domain of discourse, that is X=Y= {1,2,3,4,5}.

According the experience, we hypothesize that the major premise and the minor premise are

$\underset{\sim}{A}$ ="low temperature"=$\{\frac{1}{1}+\frac{0.7}{2}+\frac{0.4}{3}+\frac{0}{4}+\frac{0}{5}\}$,

$\underset{\sim}{B}$ ="high voltage"= $\{\frac{0}{1}+\frac{0}{2}+\frac{0.4}{3}+\frac{0.7}{4}+\frac{1}{5}\}$,

A' ="a little low temperature"= $\{\frac{1}{1}+\frac{0.6}{2}+\frac{0.4}{3}+\frac{0.2}{4}+\frac{0}{5}\}$

First, we resolve the implication $\underset{\sim}{R}$ using above two formulas we presented. For instance, we use R_B , then

$$r_{xy} = \mu_B(y) = \begin{pmatrix} 0 & 0 & 0.4 & 0.7 & 1 \\ 0 & 0 & 0.4 & 0.7 & 1 \\ 0 & 0 & 0.4 & 0.7 & 1 \\ 0 & 0 & 0.4 & 0.7 & 1 \\ 0 & 0 & 0.4 & 0.7 & 1 \end{pmatrix}$$

$$B' = \underset{\sim}{A} \circ R = (1 \quad 0.6 \quad 0.4 \quad 0.2 \quad 0) \circ \begin{pmatrix} 0 & 0 & 0.4 & 0.7 & 1 \\ 0 & 0 & 0.4 & 0.7 & 1 \\ 0 & 0 & 0.4 & 0.7 & 1 \\ 0 & 0 & 0.4 & 0.7 & 1 \\ 0 & 0 & 0.4 & 0.7 & 1 \end{pmatrix}$$

= (0 0 0.4 0.4 1)

That is $B' = \frac{0}{1}+\frac{0}{2}+\frac{0.4}{3}+\frac{0.4}{4}+\frac{1}{5}$, this result indicates that the voltage could be inflicted with a little high level.

For the second formula R_X ,

$$r_{xy} = \int_{X\times Y} [\mu_A(x)\wedge\mu_B(y)]\vee[(1-\mu_A(x))\wedge(1-\mu_B(y))]$$

We can calculate the implication is

$$r_{xy} = \begin{pmatrix} 0 & 0 & 0.4 & 0.7 & 1 \\ 0.3 & 0.3 & 0.4 & 0.7 & 0.7 \\ 0.6 & 0.6 & 0.6 & 0.6 & 0.4 \\ 1 & 1 & 0.6 & 0.3 & 0 \\ 1 & 1 & 0.6 & 0.3 & 0 \end{pmatrix}$$

$$\underset{\sim}{B} = \underset{\sim}{A} \circ R = (0.4 \;\; 0.4 \;\; 0.4 \;\; 0.7 \;\; 1)$$

That is $\underset{\sim}{B} = \frac{0.4}{1}+\frac{0.4}{2}+\frac{0.4}{3}+\frac{0.7}{4}+\frac{1}{5}$, this result indicates that the voltage could be inflicted with a little high level.

In the literature[14], some other fuzzy implication methods are also introduced. For instance,

$R_S = \underset{\sim}{A} \to \underset{\sim}{B} = \int_{X\times Y} (\mu_A(x) > \mu_B(y))/(x,y)$, In this method, if $\mu_A(x) \le \mu_B(y)$, then

$\mu_A(x) > \mu_B(y)$ =1, else $\mu_A(x) > \mu_B(y)$ =0.

Another implication method is

$R_\Delta = \underset{\sim}{A} \to \underset{\sim}{B} = \int_{X\times Y} (\mu_A(x) >> \mu_B(y))/(x,y)$,

here if $\mu_A(x) \le \mu_B(y)$, then $\mu_A(x) >> \mu_B(y)$ =1, else $\mu_A(x) >> \mu_B(y) = \mu_B(y)/\mu_A(x)$.

Table4. Close degree

fuzzy implication operation methods	Euclidean close degree	Hamming close degree
$R_c = \underset{\sim}{A} \to \underset{\sim}{B} = \int_{X\times Y} (\mu_A(x)\wedge\mu_B(y))/(x,y)$	0.90	0.94
$R_m = \underset{\sim}{A} \to \underset{\sim}{B} = \int_{X\times Y} (\mu_A(x)\wedge\mu_B(y))\vee(1-\mu_A(x))/(x,y)$	0.7951	0.86
$R_b = \underset{\sim}{A} \to \underset{\sim}{B} = \int_{X\times Y} (1-\mu_A(x))\vee\mu_B(y)/(x,y)$	0.7951	0.86
$R_p = \underset{\sim}{A} \to \underset{\sim}{B} = \int_{X\times Y} \mu_A(x)\mu_B(y)/(x,y)$	0.90	0.94
$R_S = \underset{\sim}{A} \to \underset{\sim}{B} = \int_{X\times Y} [\mu_A(x) > \mu_B(y)]/(x,y)$	0.9106	0.96
$R_\Delta = \underset{\sim}{A} \to \underset{\sim}{B} = \int_{X\times Y} (\mu_A(x) >> \mu_B(y))/(x,y)$	0.9097	0.9457
$R_B = \underset{\sim}{A} \to \underset{\sim}{B} = \int_{X\times Y} (\mu_A(x)\wedge\mu_B(y))\vee((1-\mu_A(x))\wedge\mu_B(y))/(x,y) = \int_{X\times Y} \mu_B(y)/(x,y)$	0.8735	0.92
$R_X = \underset{\sim}{A} \to \underset{\sim}{B} = \int_{X\times Y} [\mu_A(x)\wedge\mu_B(y)]\vee[(1-\mu_A(x))\wedge(1-\mu_B(y))]/(x,y)$	0.7951	0.86

Usually a fuzzy implication method is chosen according to the actual problem we need to resolve. However, we can compare these fuzzy implication formulas using a close degree method. For example, according to the experience, we define B'="a little high voltage"= $\{ \frac{0}{1}+\frac{0.2}{2}+\frac{0.4}{3}+\frac{0.6}{4}+\frac{1}{5} \}$, then we calculate the close degree of B' with those results gotten by those fuzzy implication methods. The maximum close degree is the best method for this problem.

The fuzzy distance, such as Euclidean distance and Hamming distance, could be chosen as the measurement of the close degree. For example, according to the Euclid distance that $d(\underset{\sim}{A},\underset{\sim}{B})=(\sum_{i=1}^{n}\left|\underset{\sim}{A}(x_i)-\underset{\sim}{B}(x_i)\right|^2)^{1/2}$, we get the distance between B' and the result by R_B is 0.8735.

In the table 4, we list our results upon above example using Euclidean close degree and Hamming close degree. Here suppose standard B'="a little high voltage"= $\{ \frac{0}{1}+\frac{0.2}{2}+\frac{0.4}{3}+\frac{0.6}{4}+\frac{1}{5} \}$ .The results show that when solving this problem, R_X has the same effect with R_m and R_b , and R_B has a better effect than R_m R_b and R_X .

6. Conclusion and Future Work

The main contribution of this paper is investigating fuzzy implication methods of fuzzy logic from the perspective of novel expanded two valued logical implications. From simplifying formulas of expanded two valued logical implication, we present two novel fuzzy implication operation methods, which match the second case and the third case in the expanded truth table. A practical example proves their soundness and reasonability.

The research approach, which proceeds from traditional two valued logic implication formulas to deducing fuzzy implication formulas, is clear and intuitionistic.

Fuzzy implication is a basis problem of approximate reasoning. We are constantly studying on the approximate reasoning and applying it into intelligent control in the industry. The approximate reasoning in this paper is a simplest mode. In the future work, we will further investigate the choice of fuzzy implication methods in the multi dimensional, multiple and multi output mode of approximate reasoning.

In addition, Karnaugh maps as a simplifying tool did a great favor in our research. In the future work, we will also utilize this tool to study the multi valued fuzzy logic.

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